Mapping of Shape Invariant Potentials by the Point Canonical Transformation

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Abstract In this paper by using the method of point canonical transformation we find that the Coulomb and Kratzer potentials can be mapped to the Morse potential. Then we show that the Pöschl-Teller potential type I belongs to the same subclass of shape invariant potentials as Hulthén potential. Also we show that the shape-invariant algebra for Coulomb, Kratzer, and Morse potentials is $SU(1, 1)$, while the shape-invariant algebra for Pöschl-Teller type I and Hulthén is $SU(2)$.

Keywords Point canonical transformation · Supersymmetric quantum mechanics · Shape-invariant

1 Introduction

It is known that all analytically solvable potentials in quantum mechanics have the property of shape invariance [1]. In fact shape invariance is an integrability condition, however, one should emphasize that shape invariance is not the most general integrability condition as not all exactly solvable potentials seem to be shape invariance to [2, 3]. An interesting feature of supersymmetric quantum mechanics is that for a shape invariant system [4, 5] the entire spectrum can be determined algebraically without ever referring to underlying differential equations.

In this paper we briefly describe supersymmetric quantum mechanics, then by using the method of point canonical transformation we find that the Coulomb and Kratzer potentials can be mapped to the Morse potential [6]. The Kratzer potential [7] we consider in this paper has played an important role in the history of the molecular and quantum chemistry and it has been so far extensively used to describe the molecular structure and interactions [11].

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After that we show that the Pöschl-Teller potential type I belongs to the same subclass of shape invariant potentials as Hulthén potential. The Hulthén potential [8, 9] is one of the important short-range potentials in physics. This potential is a special case of the Eckart potential [10] which has been widely used in several branches of physics and its bound-state and scattering properties have been investigated by a variety of techniques.

2 Supersymmetry Quantum Mechanics and Shape Invariance

According to the factorization method [14, 15], the quantum mechanical Hamiltonian, after subtracting the ground energy, is written as the product of an operator $\hat{A}$ and its Hermitian conjugate, $\hat{A}^{\dagger}$

$$\hat{H} - E_0 = \hat{A}^{\dagger}\hat{A} \tag{1}$$

where E_0 is the ground state energy, and $\hat{A}$, $\hat{A}^{\dagger}$ are given by

$$\hat{A} = W(x) + \frac{i}{\sqrt{2m}}\hat{p}, \tag{2}$$

$$\hat{A}^{\dagger} = W(x) - \frac{i}{\sqrt{2m}}\hat{p}. \tag{3}$$

By definition (1) the ground state wave function satisfies the following condition

$$\hat{A}|\psi_0\rangle = 0 \tag{4}$$

Since the ground-state wave function $\psi_0(x)$ for a bound state has no node, it can be written as

$$\psi_0(x) = e^{-\frac{\sqrt{2m}}{\hbar}\int W(x)\,dx}. \tag{5}$$

Using (4) we have the following supersymmetric partner Hamiltonian

$$\hat{H}_1 = \hat{A}^{\dagger}\hat{A}, \qquad \hat{H}_2 = \hat{A}\hat{A}^{\dagger}. \tag{6}$$

The corresponding potentials are given as

$$V_1 = W^2(x) - \frac{\hbar}{\sqrt{2m}}\frac{dW(x)}{dx}, \tag{7}$$

$$V_2 = W^2(x) + \frac{\hbar}{\sqrt{2m}}\frac{dW(x)}{dx}. \tag{8}$$

The Hamiltonian in (1) is called shape-invariant [3] if the following condition is satisfied:

$$\hat{A}(a_1)\hat{A}^{\dagger}(a_1) = \hat{A}^{\dagger}(a_2)\hat{A}(a_2) + R(a_{a_1}) \tag{9}$$

where a_1, and a_2 represent the parameters of the Hamiltonian. One can rewrite the above condition in term of the partner potentials as:

$$V_2(x, a_1) = V_1(x, a_2) + R(a_1) \tag{10}$$

shape-invariant problem was formulated in algebraic terms in [16]. We assume that replacing a_1 by a_2 in a given operator can be achieved with a similarity transformation

$$\hat{T}(a_1)\mathcal{O}(a_1)\hat{T}^{-1}(a_1) = \mathcal{O}(a_2). \tag{11}$$

There are two classes of shape-invariant potentials. For the first class the parameters a_1 and a_2 of the two supersymmetric parameters are related to each other by translation [2, 17]

$$a_2 = a_1 + \eta. \tag{12}$$

For the second class, the parameters a_1 and a_2 are related to each other by scaling [12, 13]

$$a_2 = q a_1. \tag{13}$$

For the first class the operator $\hat{T}(a_1)$ of (11) is given by

$$\hat{T}(a_1) = e^{\eta \frac{\partial}{\partial a_1}}, \qquad \hat{T}^{-1}(a_1) = \hat{T}^{\dagger}(a_1). \tag{14}$$

In the second class, the similarity transformation (11) is given by following operator

$$\hat{S}(a_1) = e^{\ln q \, a_1 \frac{\partial}{\partial a_1}}, \qquad \hat{S}^{-1}(a_1) = \hat{S}^{\dagger}(a_1). \tag{15}$$

By introducing new operators

$$\hat{B}_+ = \hat{A}^{\dagger}(a_1)\hat{T}(a_1), \qquad \hat{B}_- = \hat{B}_+^{\dagger} = \hat{T}^{\dagger}(a_1)\hat{A}(a_1) \tag{16}$$

the Hamiltonian can be rewritten as

$$\hat{H} - E_0 = \hat{A}^{\dagger}\hat{A} = \hat{B}_+\hat{B}_-. \tag{17}$$

Using (9) and (16), one can obtain following commutation relation

$$[\hat{B}_-, \hat{B}_+] = R(a_0) \tag{18}$$

where

$$a_n = a_0 + n\,\eta \quad \text{or} \quad a_n = q^n \, a_0 \tag{19}$$

also following identities

$$R(a_n) = \hat{T}(a_1)R(a_{n-1})\hat{T}^{\dagger}(a_1), \qquad R(a_n) = \hat{S}(a_1)R(a_{n-1})\hat{S}^{\dagger}(a_1) \tag{20}$$

valid for any n. By using (16, 20) we can establish the commutation relations

$$[\hat{H}, \hat{B}_+^n] = (R(a_1) + R(a_2) + \cdots + R(a_n))\hat{B}_+^n, \tag{21}$$

$$[\hat{H}, \hat{B}_-^n] = -\hat{B}_-^n(R(a_1) + R(a_2) + \cdots + R(a_n)) \tag{22}$$

means that, $B_+^n|\psi_0\rangle$ is an eigenstate of the Hamiltonian with the eigenvalue $R(a_1) + R(a_2) + \cdots + R(a_n)$. The normalized eigenstate is

$$|\psi_n\rangle = \frac{1}{\sqrt{R(a_1) + \cdots + R(a_n)}} \hat{B}_+ \times \cdots \times \frac{1}{\sqrt{R(a_1) + R(a_2)}} \hat{B}_+ \times \frac{1}{\sqrt{R(a_1)}} \hat{B}_+|\psi_0\rangle. \tag{23}$$

In addition to the oscillatorlike commutation relations (21) one gets the commutation relations

$$[\hat{B}_+, R(a_0)] = \{R(a_1) - R(a_0)\}\hat{B}_+, \tag{24}$$

$$[\hat{B}_+, [\hat{B}_+, R(a_0)]] = (\{R(a_2) - R(a_1)\} - \{R(a_1) - R(a_0)\})\hat{B}_+^2 \tag{25}$$

and so on.

3 Mapping of Kratzer and Coulomb Potentials to the Morse Potential

Consider the following potential (we are using units with $\hbar = 1$, $2m = 1$).

$$V(x) = -\frac{\alpha}{x} + \frac{\beta}{x^2} + \gamma. \tag{26}$$

If we take $\alpha = \beta = 1$ and $\gamma = 0$, we obtain the Kratzer potential as

$$V(x) = -\left(\frac{1}{x} - \frac{1}{x^2}\right) \tag{27}$$

in another case we take $\alpha = e^2$, $\beta = l(l+1)$ and $\gamma = \frac{e^4}{4(l+1)^2}$, in this case the potential (26) is as

$$V(x) = -\frac{e^2}{x} + \frac{l(l+1)}{x^2} + \frac{e^4}{4(l+1)^2} \tag{28}$$

which is equivalent with Coulomb potential in 3-dimension Schrödinger equation in spherical coordinates.

At first, we briefly review the method of mapping of shape-invariant under point canonical transformation. For given potential of (26), one can write the Schrödinger equation as

$$\left\{-\frac{d^2}{dx^2} + V(\alpha_i; x) - E(\alpha_i)\right\}\psi(\alpha_i; x) = 0 \tag{29}$$

here α_i are the set of parameters of given potential (26). Under a point canonical transformation, as following

$$x := f(z), \qquad \psi(\alpha_i, x) := g(z)\widetilde{\psi}(\widetilde{\alpha}_i; z) \tag{30}$$

the Schrödinger equation (29) is transformed into

$$\left\{-\frac{d^2}{dz^2} + \left(\frac{f''}{f'} - 2\frac{g'}{g}\right)\frac{d}{dz} + \left(\frac{g'}{g}\frac{f''}{f'} - \frac{g''}{g}\right)f'^2(V(\alpha_i; f(z)) - E(\alpha_i))\right\}\widetilde{\psi}(\widetilde{\alpha}_i; z) = 0 \tag{31}$$

or in the familiar form as

$$\left\{-\frac{d^2}{dz^2} + \widetilde{V}(\widetilde{\alpha}_i; z) - \widetilde{E}(\widetilde{\alpha}_i)\right\}\widetilde{\psi}(\widetilde{\alpha}_i; z) = 0 \tag{32}$$

in which α_i represent set of parameters of the transformed potential, and the prime denotes differential with respect to the variable z. To remove the first-derivative term from (31), one requires

$$g(z) = C\sqrt{f'(z)}. \tag{33}$$

Using (33) and comparing (30, 31) we obtain

$$\widetilde{V}(\widetilde{\alpha}_i; z) - \widetilde{E}(\widetilde{\alpha}_i) = f'^2\{V(\alpha_i; f(z)) - E(\alpha_i)\} + \frac{1}{2}\left\{\frac{3}{2}\left(\frac{f''}{f'}\right)^2 - \frac{f'''}{f'}\right\}. \tag{34}$$

By substitution (26) into (34), we have

$$\widetilde{V}(\widetilde{\alpha}_i; z) - \widetilde{E}(\widetilde{\alpha}_i) = f'^2\left(-\frac{\alpha}{f} + \frac{\beta}{f^2} + \gamma - E\right) + \frac{1}{2}\left\{\frac{3}{2}\left(\frac{f''}{f'}\right)^2 - \frac{f'''}{f'}\right\}. \tag{35}$$

We consider

$$f(z) = e^{-z} \tag{36}$$

by this selection, one can define a point canonical transformation as

$$f(z) = e^{-z}, \qquad g(z) = e^{-\frac{z}{2}} \tag{37}$$

with above transformation, we can rewrite (35) as

$$\widetilde{V}(\widetilde{\alpha}_i; z) - \widetilde{E}(\widetilde{\alpha}_i) = \left(\gamma - E + \frac{3}{4}\right)e^{-2z} - \alpha\, e^{-z} + \left(\beta - \frac{1}{2}\right) \tag{38}$$

which is like to Morse potential. In other words, by acting the point canonical transformation (37) on the potential of (26), that can explain the Kratzer and Coulomb potentials, we obtain the Morse potential. In this situation we are looking for this potential's algebra. For Morse potential

$$\widetilde{V}(z) = e^{-2z} - 2be^{-z} \tag{39}$$

the superpotential is

$$\widetilde{W}(z; a_n) = a_n - e^{-z}. \tag{40}$$

Therefore the reminder in (10) is given by

$$R(a_n) = 2(a_n - 1) \tag{41}$$

where

$$a_n = b - \left(n + \frac{1}{2}\right). \tag{42}$$

One can use (19)

$$R(a_n) - R(a_{n-1}) = -2. \tag{43}$$

Therefore, the commutation relation of (25) will vanish. Now, we define the following dimensionless operators

$$\hat{K}_0 := \frac{1}{4}R(a_0) \tag{44}$$

and

$$\hat{K}_\pm := \frac{1}{\sqrt{2}}\hat{B}_\pm \tag{45}$$

where $\hat{B}_{\pm}$ has defined by (16). One can find that the shape-invariant algebra for these potentials is $SU(1, 1)$

$$[\hat{K}_+, \hat{K}_-] = \frac{1}{2}[\hat{B}_+, \hat{B}_-] = 2\left(-\frac{1}{4}R(a_0)\right) = -2\hat{K}_0, \tag{46}$$

$$[\hat{K}_0, \hat{K}_{\pm}] = \frac{1}{4\sqrt{2}}[R(a_0), \hat{B}_{\pm}] = \pm\left(\frac{4}{4\sqrt{2}}\hat{B}_{\pm}\right) = \pm\hat{K}_{\pm}. \tag{47}$$

4 Mapping of Hulthén Potential into Pöschl-Teller Potential 1

The Hulthén potential has the following form (we are using units with $\hbar = 1$, $2m = 1$)

$$V(r) = -\frac{e^{-r}}{1 - e^{-r}} \tag{48}$$

for mapping of this potential, we consider

$$f(z) = -2\ln[\cos z] \tag{49}$$

by this selection, one can define a point canonical transformation as

$$f(z) = -2\ln[\cos z]$$
$$g(z) = \sqrt{-2\ln[\cos z]} \tag{50}$$

with above transformation, we can rewrite (35) as

$$\widetilde{V}(\widetilde{\alpha}_i; z) - \widetilde{E}(\widetilde{\alpha}_i) = 4(E - 1) - \frac{1}{4}(1 + 16E)\sec^2 z + \frac{3}{4}\csc^2 z \tag{51}$$

which is like to Pöschl-Teller potential 1. For Pöschl-Teller potential 1

$$\widetilde{V}(z) = -(A + B)^2 + A(A - 1)\sec^2 z + B(B - 1)\csc^2 z \tag{52}$$

the energy eigenstates are given by

$$\widetilde{E}_n = (A + B + 2n)^2 - (A + B)^2. \tag{53}$$

Therefore the reminder in (10) is given by (one can find $R(a_n) = \widetilde{E}_n - \widetilde{E}_{n-1}$)

$$R(a_n) = 4(2n + A + B - 1). \tag{54}$$

One can use (19)

$$R(a_n) - R(a_{n-1}) = 8. \tag{55}$$

Therefore, the commutation relation of (25) will vanish. Now, we define the following dimensionless operators

$$\hat{K}_0 := \frac{-1}{8}R(a_0) \tag{56}$$

and

$$\hat{K}_\pm := \frac{1}{2}\hat{B}_\pm \qquad (57)$$

where $\hat{B}_\pm$ has defined by (16). One can find that the shape-invariant algebra for these potentials is $SU(2)$

$$[\hat{K}_+, \hat{K}_-] = \frac{1}{4}[\hat{B}_+, \hat{B}_-] = 2\left(-\frac{1}{8}R(a_0)\right) = 2\hat{K}_0, \qquad (58)$$

$$[\hat{K}_0, \hat{K}_\pm] = \frac{-1}{16}[R(a_0), \hat{B}_\pm] = \pm\left(\frac{1}{2}\hat{B}_\pm\right) = \pm\hat{K}_\pm. \qquad (59)$$

5 Conclusion

For exactly solvable potentials of nonrelativistic quantum mechanics, eigenvalues and eigenvectors can be derived using well known methods of supersymmetric quantum mechanics. In this paper the Schrödinger equation with some potentials (Coulomb, Kratzer, with Morse and Pöschl-Teller type I with Hulthén) has been studied and we have shown that such potentials can be easily inter-related among themselves within the framework of point canonical coordinate transformations as the corresponding eigenvalues may be written down in a closed form algebraically using the well known results for the shape invariant potentials. Also we have shown that the shape-invariant algebra for Coulomb, Kratzer, and Morse potentials is $SU(1, 1)$, while the shape-invariant algebra for Pöschl-Teller type I and Hulthén is $SU(2)$. We must mention that the Morse potential is also related with the $SU(2)$ group except for $SU(1, 1)$ one, to see the $SU(2)$ group approach refer to [18].

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